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HEREDITARY PROPERTIES OF THE CLASS OF CLOSED SETS OF UNIQUENESS FOR TRIGONOMETRIC SERIES

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ABSTRACT

It is shown that the σ-ideal U_0 of closed sets of extended uniqueness in $\mathbf{T}$ is hereditarily non-Borel, i.e. every "non-trivial" σ-ideal of closed sets $I \subseteq U_0$ is non-Borel. This implies both the result of Solovay, Kaufman that both U_0 and U (the σ-ideal of closed sets of uniqueness) are not Borel as well as the theorem of Debs-Saint Raymond that every Borel subset of $\mathbf{T}$ of extended uniqueness is of the first category. A further extension to ideals contained in U_0 is given.

§1. Introduction

Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be the unit circle. A subset $P \subseteq \mathbf{T}$ is called a set of *uniqueness* if every trigonometric series converging to 0 on $\mathbf{T} - P$ is identically 0 and is called a set of *extended uniqueness* if this uniqueness property holds for the trigonometric series which are Fourier-Stieltjes series $\sum \hat{\mu}(n)e^{inx}$ of (finite Borel) measures on $\mathbf{T}$.

Let $K(\mathbf{T})$ be the compact, metric space of closed subsets of $\mathbf{T}$ with the Hausdorff metric. We denote by U the class of $E \in K(\mathbf{T})$ which are sets of uniqueness and by U_0 the class of $E \in K(\mathbf{T})$ which are sets of extended uniqueness. Let also $M = K(\mathbf{T}) - U$ and $M_0 = K(\mathbf{T}) - U_0$ be the classes of closed sets of *multiplicity* and *restricted multiplicity,* resp.

It has been shown by Solovay (see, e.g., [6]) and Kaufman [4] that the classes U, U_0 are complete coanalytic ($\mathbf{\Pi}^1_1$) and thus, in particular, non-Borel in the space $K(\mathbf{T})$. On the other hand, Debs-Saint Raymond [2] have shown that every Borel set of extended uniqueness is of the first category, thereby solving N. Bary's Cat-

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egory Problem (see [1]). For more on these recent results on the theory of sets of uniqueness and its relations with descriptive set theory, see [6].

The main purpose of this paper is to prove a rather surprising hereditary definability property of the class U_0, which in particular implies both the above results.

Recall that a *σ-ideal of closed sets* is a class $I \subseteq K(\mathbf{T})$ such that (i) $(E, F \in K(\mathbf{T});\ E \subseteq F \in I) \Rightarrow E \in I$, i.e. I is *hereditary*, and (ii) $(E, E_n \in K(\mathbf{T});\ E = \bigcup_n E_n,\ E_n \in I) \Rightarrow E \in I$, i.e. I is closed under countable unions, which are closed. We denote below by M_0^p the class of $E \in K(\mathbf{T})$ of *pure restricted multiplicity*, i.e. those for which $\overline{E \cap V} \in M_0$ for every open interval V with $E \cap V \neq \emptyset$. For $E \in K(\mathbf{T})$, $K(E) = \{F \in K(\mathbf{T}) : F \subseteq E\}$.

THEOREM 1. *Let I be a σ-ideal of closed sets, $I \subseteq U_0$. Then I is not analytic ($\mathbf{\Sigma}_1^1$), provided it satisfies the following non-triviality condition: For some $E \in M_0^p$, $E \neq \emptyset$, $I \cap K(E)$ is dense in $K(E)$. (For example, this is satisfied if $\{x\} \in I$ for all $x \in D$, D some dense subset of E.)*

In particular, if $K_\omega(\mathbf{T})$ is the σ-ideal of countable closed subsets of $\mathbf{T}$, there is no $\mathbf{\Sigma}_1^1$ σ-ideal I with $K_\omega(\mathbf{T}) \subseteq I \subseteq U_0$. So if such an I is $\mathbf{\Pi}_1^1$, then by the Dichotomy Theorem of [7], I is complete $\mathbf{\Pi}_1^1$. Thus U_0 is *hereditary* $\mathbf{\Pi}_1^1$-complete.

Theorem 1 clearly includes the theorem of Solovay and Kaufman. To see that it implies also the Debs–Saint Raymond result, let P be a Borel set of extended uniqueness and assume P is of the second category towards a contradiction. Then let V be an open interval and $G \subseteq V$ a dense G_δ in V with $G \subseteq P$. Then $K(G) = \{E \in K(\mathbf{T}) : E \subseteq G\}$ is G_δ in $K(\mathbf{T})$ and $K(G) \subseteq U_0$. But for $E = \overline{V}$, if $D = G$ then D is dense in E and for all $x \in D$, $\{x\} \in I$, so this violates Theorem 1.

The proof of Theorem 1 combines methods of Körner (see [3], p. 118) and Kaufman [5] (see also [6], p. 239) along with results of Kechris–Louveau [6], p. 274.

One can also use the proof of Theorem 1 to show the following extension.

THEOREM 2. *Let $I \subseteq U_0$ be hereditary G_δ and assume $I \cap K(E)$ is dense in $K(E)$ for some $E \neq \emptyset$, $E \in M_0^p$. Let I_f be the class of finite unions of sets from I. Then there is no G_δ set G with*

$$I_f \subseteq G \subseteq U_0.$$

By the Hurewicz-type theorem proved in [7] (see also [6], p. 133) this is equivalent to saying that there is a homeomorphic copy F of the Cantor space $2^{\mathbf{N}}$ with $F \subseteq I_f \cup M_0$ and $F \cap I_f$ countable dense in F. For example, this implies that if $Q \subseteq 2^{\mathbf{N}}$ is countable dense, there is continuous $f : 2^{\mathbf{N}} \to K(T)$ such that

$$x \in Q \Rightarrow f(x) \text{ is a finite union of Kronecker sets,}$$

$$x \notin Q \Rightarrow f(x) \text{ is an } M_0\text{-set.}$$

(Recall that a *Kronecker* set is a closed set $E \in K(\mathbf{T})$ such that for every continuous $f: E \to \mathbf{T}$ and every $\epsilon > 0$ there is $n \in \mathbf{Z}$ with $\|e^{inx} - f(x)\| < \epsilon$, $\forall x \in E$. The class of Kronecker sets is a hereditary dense G_δ in $K(\mathbf{T})$ (see [6], p. 337).

It follows also from Theorem 2 that there is no G_δ *ideal* $I \subseteq U_0$ which is dense in $K(E)$ for some $E \in M_0^p$, $E \neq \varnothing$. (An *ideal* is a hereditary, closed under finite unions class.) This is not, however, a real strengthening of Theorem 1 in view of the following general result.

THEOREM 3. (Dougherty–Kechris, Louveau). *Let E be a compact, metrizable space. If $I \subseteq K(E)$ is a G_δ ideal, then I is a σ-ideal.*

We conclude with the following interesting problem, an affirmative answer to which would give also a different proof of Theorem 1:

Let E be compact, metrizable and $I \subseteq K(E)$ a G_δ σ-ideal of closed sets on E. Assume I contains all singletons or just all singletons in a dense subset of E. Is there a dense G_δ set $G \subseteq E$ such that $K(G) \subseteq I$?

§2. Proof of Theorem 1

The key to the proof is the following lemma which might be of interest in its own sake. Its proof uses methods of Körner (see [3], p. 118) and Kaufman ([5], or see [6], p. 239).

Below, a *Rajchman measure* on $\mathbf{T}$ is a measure μ with $\hat{\mu}(n) \to 0$ as $|n| \to \infty$. The closed support of a measure is denoted by $\mathrm{supp}(\mu)$. Finally, $\|\ \|_{PM}$ denotes the pseudomeasure norm, i.e. $\|\rho\|_{PM} = \sup\{|\hat{\rho}(n)| : n \in \mathbf{Z}\}$.

LEMMA 2.1. *Let μ be a probability Rajchman measure on $\mathbf{T}$ with support* $\mathrm{supp}(\mu) = E$. *Let $I \subseteq K(E)$ be G_δ hereditary and dense in $K(E)$. Then, given $N > 0$, $\epsilon > 0$ there is a probability measure ν with* $\mathrm{supp}(\nu) = E_1 \cup \cdots \cup E_N$ *where $E_i \in I$ $(1 \le i \le N)$ and $\|\mu - \nu\|_{PM} \le (1 + \epsilon)/N$.*

PROOF. We will define probability Rajchman measures μ_k and integers n_k such that

(1) $0 < n_0 = n_1 = \cdots = n_{N-1} < n_N < n_{N+1} < \cdots$,

(2) $\mu_0 = \mu_1 = \cdots = \mu_{N-1} = \mu$,

(3) $\left(|j| \le n_{k+N-1} \vee |j| \ge n_{k+N}\right) \Rightarrow |\hat{\mu}_{k+N}(j) - \hat{\mu}_k(j)| \le \frac{1}{2}\epsilon \cdot 2^{-k-1}$,

(4) $n_k \le |j| \Rightarrow |\hat{\mu}_k(j)| < \epsilon/2$,

(5) $\text{supp}(\mu_{k+N}) \subseteq \text{supp}(\mu_k)$,

(6) $\text{supp}(\mu_{n+kN}) \in G_k$, $k = 1,2,\ldots,n = 0,\ldots,N-1$, where $I = \bigcap_{k=1}^{\infty} G_k, G_k$ decreasing and open, hereditary,

(7) $\text{supp}(\mu_k) = \bigcup_{i=1}^{P_k} A_i^{(k)}$, where $A_i^{(k)} = E \cap \overline{I_i^{(k)}}$, $\{I_i^{(k)}\}_{i=1}^{P_k}$ open intervals with $E \cap I_i^{(k)} \neq \varnothing$ and $\overline{I_i^{(k)}} \cap \overline{I_j^{(k)}} = \varnothing$, if $i \neq j$.

The construction is by induction. Assume it has been done up to $k + N - 1$ ($k = 0,1,2,\ldots$). We will construct μ_{k+N}, n_{k+N}. Let $E_k = \text{supp}(\mu_k)$.

Fix a finite set of pairwise disjoint open intervals $I_1,\ldots,I_m$ with $E_k \subseteq \bigcup_{i=1}^m \bar{I}_i$, $E_k \cap I_i \neq \varnothing$, such that if ρ,σ are Rajchman probability measures with $\text{supp}(\rho)$, $\text{supp}(\sigma) \subseteq \bigcup_{i=1}^m \bar{I}_i$ and $\rho(I_i) = \sigma(I_i)$ $(i = 1,\ldots,m)$, then

$$|j| \leq n_{k+N-1} \Rightarrow |\hat{\rho}(j) - \hat{\sigma}(j)| \leq \frac{1}{2}\epsilon \cdot 2^{-k-1}.$$

(For the reader's convenience, we explain how these intervals can be found — the argument coming from Körner's proof in [3], p. 118. By direct calculations, if ρ,σ are positive measures and $I = [t-d, t+d]$ is an interval with $\rho(I) = \sigma(I)$, then

$$\left|\int_I e^{-inx}\,d\rho(x) - \int_I e^{-inx}\,d\sigma(x)\right| = \left|\int_I (e^{-inx} - e^{-int})\,d(\rho - \sigma)(x)\right|$$
$$\leq 2\rho(I) \max_{x \in I} |e^{-inx} - e^{-int}|.$$

So one simply chooses the $I_1,\ldots,I_m$ to have sufficiently small length.)

By (7), I is dense in $K(E_k)$ as well, so let $K \in I$, $K \subseteq E_k$ be such that $K \cap I_i \neq \varnothing$, $i = 1,\ldots,m$. If $k = n + lN$ $(0 \leq n \leq N-1, l \geq 0)$, $K \in G_{l+1}$, so find open V with $K \in K(V \cap E_k) \subseteq G_{l+1}$ (this can be done since G_{l+1} is open hereditary). Let then $J_1,\ldots,J_m$ be open intervals with $\bar{J}_i \cap \bar{J}_j = \varnothing$ if $i \neq j$, $\bar{J}_i \subseteq I_i$, $\bigcup_{i=1}^m \bar{J}_i \subseteq V$ and $E_k \cap J_i \neq \varnothing$ for $i = 1,\ldots,m$. Define the probability measure μ_{k+N} to have support

$$E_k \cap \bigcup_{i=1}^m \bar{J}_i (\subseteq E_k \cap V)$$

and

$$\mu_{k+N} \upharpoonright \bar{J}_i = \mu_k \upharpoonright \bar{J}_i \cdot \frac{\mu_k(I_i)}{\mu_k(J_i)}.$$

Since μ_k is a Rajchman measure, so is μ_{k+N} (as $\mu_{k+N} \ll \mu_k$) and (7) is clearly satisfied, as well as (5), (6). Now

$$\mu_{k+N}(I_i) = \mu_k(I_i \cap \bar{J}_i) \cdot \frac{\mu_k(I_i)}{\mu_k(J_i)}$$

$$= \mu_k(J_i) \cdot \frac{\mu_k(I_i)}{\mu_k(J_i)} = \mu_k(I_i)$$

(recall that Rajchman measures are continuous), so (3) is satisfied for $|j| \leq n_{k+N-1}$. Finally, choose $n_{k+N} > n_{k+N-1}$ large enough so that the second part of (3) and also (4) for n_{k+N}, μ_{k+N} are satisfied.

Put now $\mu^n = \lim_l^{w^*} \mu_{n+l\cdot N}$, for $n = 0,1,\ldots,N-1$. Then $\operatorname{supp}(\mu^n) \subseteq \operatorname{supp}(\mu_{n+lN}) \in G_l$, for $l \geq 1$, so $\operatorname{supp}(\mu^n) \in I$. Let $\nu = (1/N)(\mu^0 + \cdots + \mu^{N-1})$. Then if $E_i' = \operatorname{supp}(\mu^{i-1})$, $i = 1,\ldots,N$, $\operatorname{supp}(\nu) \subseteq E_1' \cup \cdots \cup E_N'$, so $\operatorname{supp}(\nu) = E_1 \cup \cdots \cup E_N$, where $E_i = \operatorname{supp}(\nu) \cap E_i' \in I$.

Also, for each j, $|\hat{\mu}(j) - \hat{\nu}(j)| \leq (1/N)\sum_{k=0}^{\infty} |\hat{\mu}_{k+N}(j) - \hat{\mu}_k(j)|$. But

$$|\hat{\mu}_{k+N}(j) - \hat{\mu}_k(j)| \leq \frac{1}{2}\epsilon \cdot 2^{-k-1}, \qquad \text{if } |j| \leq n_{k+N-1} \text{ or } |j| \geq n_{k+N}$$

and (as $n_k < n_{k+N-1}$), if $n_{k+N-1} < |j| < n_{k+N}$, then $|\hat{\mu}_{k+N}(j) - \hat{\mu}_k(j)| \leq 1 + \epsilon/2$. So

$$|\hat{\mu}(j) - \hat{\nu}(j)| \leq \frac{1}{N} \cdot \left(1 + \frac{\epsilon}{2} + \sum_{k=0}^{\infty} \frac{\epsilon}{2} \cdot 2^{-k-1}\right)$$

$$= \frac{1}{N} \cdot (1 + \epsilon),$$

i.e.,

$$\|\mu - \nu\|_{PM} \leq \frac{1}{N} \cdot (1 + \epsilon). \qquad \blacksquare$$

Denote for each $E \in K(\mathbf{T})$,

$\eta_0(E) = \inf\{R(\mu) : \mu$ a probability measure whose support is contained in $E\}$,

where $R(\mu) = \overline{\lim} |\hat{\mu}(n)|$.

The following follows immediately from Lemma 2.1.

Corollary 2.2. *Let I be hereditary G_δ in $K(\mathbf{T})$ and assume I is dense in some $K(E)$, $E \in M_0^p$, $E \neq \emptyset$. Then for every $\epsilon > 0$ there is $F \in K(\mathbf{T})$, where $F = F_1 \cup \cdots \cup F_n$ with $F_i \in I$ $(i = 1,\ldots,n)$ and $\eta_0(F) < \epsilon$.*

Proof. As $E \in M_0^p$ there is a Rajchman measure μ with $\operatorname{supp}(\mu) = E$ (see [6], p. 269). Let ν be as in Lemma 2.2 and put $F = \operatorname{supp}(\nu)$. ■

Finally, we have

Theorem 2.3. *If I is hereditary G_δ in $K(\mathbf{T})$ and I is dense in some $K(E)$, where $E \in M_0^p$, $E \neq \varnothing$, then*

$$I \subseteq U_0 \Rightarrow I_\sigma \text{ is not } \mathbf{\Sigma}_1^1$$

where I_σ is the σ-ideal of closed sets generated by I.

In particular, Theorem 1.1 holds.

Proof. Since every portion $E' = \overline{E \cap V}$, V open interval, $V \cap E \neq \varnothing$ is in M_0^p and I is dense in $K(E')$, it follows from Corollary 2.2 that in each portion of E there are sets in I_σ with arbitrarily small η_0 and thus with $\eta_0 = 0$. By Theorems VI.1.6 and VIII.2.1 of [6] it follows that there are sets in I_σ of arbitrarily large U_σ-rank (see [6], p. 281). By the boundedness theorem for $\mathbf{\Pi}_1^1$-ranks (see [6], p. 148) it follows that I_σ cannot be $\mathbf{\Sigma}_1^1$.

To prove Theorem 1.1, notice that if I is as in the statement of that theorem and is $\mathbf{\Sigma}_1^1$, then by [7], I is actually G_δ, so since $I = I_\sigma$, we have a contradiction. ■

Remark. Lemma 2.1 can be viewed as an abstract version of the following result of Körner and Kaufman: For each N there is a finite union of N Kronecker sets $F = F_1 \cup \cdots \cup F_N$ so that F is independent (over the rationals) and $\eta_0(F) \leq 1/N$—recall that for a Kronecker set E, $\eta_0(E) = 1$ (see, e.g., [6], p. 338). By applying Lemma 2.1 to E, an independent M_0^p-set (which exists by a result of Rudin, see [3]) and $I \subseteq K(E)$ the class of Kronecker subsets of E, which is a G_δ hereditary dense subset of $K(E)$, one obtains the above with $\eta_0(F) \leq 1/N + \epsilon$.

§3. Proof of Theorem 2

We will base the proof on the following lemma.

Lemma 3.1. *Let $\mathfrak{J}$ be an ideal of closed sets in $K(\mathbf{T})$. Assume that for some $E \in M_0^p$, $E \neq \varnothing$ and every open V with $E \cap V \neq \varnothing$ and $\epsilon > 0$ there is $F \in \mathfrak{J}$ with $F \subseteq \overline{E \cap V}$ and $\eta_0(F) < \epsilon$. Then there is no G_δ set with $\mathfrak{J} \subseteq G \subseteq U_0$.*

From this and Corollary 2.2, one obtains immediately Theorem 2.

Proof of Lemma 3.1. We will need first the following sublemma.

LEMMA A. *Let $G \subseteq K(\mathbf{T})$ be hereditary G_δ, say $G = \bigcap_n G_n$, $G_n \supseteq G_{n+1}$ where G_n is open, hereditary in $K(\mathbf{T})$. Assume $E \in M_0^p$, $E \neq \emptyset$ and each G_n has the following density property.*

($*$) *For every Rajchman probability measure μ supported by E, for every $\epsilon > 0$ and every open V such that* $\mathrm{supp}(\mu) \subseteq V$, *there is a Rajchman probability measure ν with* $\mathrm{supp}(\nu) \subseteq E \cap V$, $\mathrm{supp}(\nu) \in G_n$ *and* $\|\mu - \nu\|_{PM} < \epsilon$.

Then G has also the same density property ($*$). In particular, G contains an M_0-set.

PROOF. Fix μ, V, ϵ as in ($*$). Find a Rajchman probability measure μ_1 with $\mathrm{supp}(\mu_1) \subseteq E \cap V_0$, where $V_0 \subseteq \overline{V}_0 \subseteq V$, V_0 open, $\|\mu_1 - \mu\|_{PM} < \epsilon/2$ and $\mathrm{supp}(\mu_1) \in G_1$. Since G_1 is hereditary open, find open V_1 such that $\mathrm{supp}(\mu_1) \subseteq V_1$ and $K(\overline{V}_1) \subseteq G_1$. Thus $\mathrm{supp}(\mu_1) \subseteq V_0 \cap V_1 \cap E$. Let now μ_2 be a Rajchman probability measure with $\mathrm{supp}(\mu_2) \subseteq V_0 \cap V_1 \cap E$, $\|\mu_2 - \mu_1\|_{PM} < \epsilon/4$ and $\mathrm{supp}(\mu_2) \subseteq V_2$ with $K(\overline{V}_2) \subseteq G_2$. Then $\mathrm{supp}(\mu_2) \subseteq V_0 \cap V_1 \cap V_2 \cap E$, etc. Clearly, $\mu_n \to \nu$ in PM, where ν is a Rajchman probability measure. Also, $\mathrm{supp}(\mu_n) \subseteq \overline{V}_0 \cap E$, thus $\mathrm{supp}(\nu) \subseteq V \cap E$ and $\|\mu - \nu\|_{PM} < \epsilon$. Finally, $\mathrm{supp}(\mu_n) \subseteq V_l$ if $n \geq l$, so $\mathrm{supp}(\nu) \subseteq \overline{V}_l$ for all l, so $\mathrm{supp}(\nu) \in G_l$ for all l, i.e. $\mathrm{supp}(\nu) \in G$. ■

To prove now Lemma 3.1, assume G is G_δ, $\mathcal{I} \subseteq G \subseteq U_0$ towards a contradiction. We can assume G is hereditary, otherwise replace it by $G' = \{F \in K(\mathbf{T}) : \forall F' \in K(\mathbf{T})(F' \subseteq F \Rightarrow F' \in G\}$, which is also G_δ. Write $G = \bigcap_n G_n$ with G_n open hereditary, $G_n \supseteq G_{n+1}$. It is enough to verify ($*$) of Lemma A for each G_n. For that we need the following

LEMMA B. *Let $E \in M_0^p$, $E \neq \emptyset$. Let μ be a probability measure with* $\mathrm{supp}(\mu) \subseteq E \cap V$, *$V$ open. If $R(\mu) < \epsilon$ then there is a Rajchman probability measure ν with* $\|\mu - \nu\|_{PM} < \epsilon$ *and* $\mathrm{supp}(\nu) \subseteq E \cap V$.

PROOF. Denote by P the set of Rajchman probability measures with support contained in $E \cap V$. Then, clearly, μ is in the weak*-closure of P. Since $R(\mu) < \epsilon$, an iterating-and-averaging argument as in [6], p. 276 shows that there is probability Rajchman measure ν with $\mathrm{supp}(\nu) \subseteq E \cap V$ and $\|\mu - \nu\|_{PM} < \epsilon$. ■

To verify ($*$) for each G_n, it is enough to prove instead:

(+) If μ is a probability Rajchman measure with $\mathrm{supp}(\mu) \subseteq E \cap V$, V open and $\epsilon > 0$, there is ν' a (*not* necessarily Rajchman) probability measure with $\mathrm{supp}(\nu') \subseteq V \cap E$, $\mathrm{supp}(\nu') \in G_n$ and $\|\mu - \nu'\|_{PM} < \epsilon/2$.

Because then $\mathrm{supp}(\nu') \in K(V \cap V' \cap E)$, where V' is open and $K(V') \subseteq$

G_n. Then by Lemma B, since $R(\nu') < \epsilon/2$, there is a probability Rajchman measure ν with $\|\nu - \nu'\|_{PM} < \epsilon/2$ and $\mathrm{supp}(\nu) \subseteq V \cap V' \cap E$, so $\|\mu - \nu\|_{PM} < \epsilon$, $\mathrm{supp}(\nu) \subseteq V \cap E$ and $\mathrm{supp}(\nu) \in G_n$.

To verify (+): Notice that the probability measures ν' with support in $\mathcal{J} \cap K(E \cap V)\,(\subseteq G_n \cap K(E \cap V))$ and with $R(\nu') < \epsilon/4$ form a convex set, say C. By the hypothesis of Lemma 3.1 on $\mathcal{J}$ they are weak*-dense among all probability measures with support in $K(E \cap V)$. By an iterating-and-averaging argument as in [6], p. 276 it follows that there is $\nu' \in C$ with $\|\mu - \nu'\|_{PM} < \epsilon/2$.

§4. Proof of Theorem 3

Let d be the metric on E. For $K, L \in K(E)$ let

$$\rho(K,L) = \sup\{d(x,L) : x \in K\}.$$

(This is *not* the metric of $K(E)$.)

LEMMA 4.1. *Let $I \subseteq K(E)$ be closed under finite unions. Let $K_n \in I$, $K \in I$ and $\rho(K_n,K) \to 0$ as $n \to \infty$. Then $L = K \cup (\bigcup_n K_n)$ is closed and $L \in I$.*

PROOF. That L is closed is easy. We will show now that $L \in I$. For $S \subseteq \mathbf{N}$ let

$$F(S) = K \cup \left(\bigcup_{n \in S} K_n\right).$$

Again, $F(S)$ is closed for all $S \subseteq \mathbf{N}$. Identifying $S \subseteq \mathbf{N}$ with its characteristic function, we claim that

$$F : 2^{\mathbf{N}} \to K(E)$$

is continuous. Indeed, if $S \restriction m = T \restriction m$ we have

$$F(S) = K \cup \left(\bigcup_{\substack{n<m \\ n \in S}} K_n\right) \cup \left(\bigcup_{\substack{n \geq m \\ n \in S}} K_n\right),$$

$$F(T) = K \cup \left(\bigcup_{\substack{n<m \\ n \in T}} K_n\right) \cup \left(\bigcup_{\substack{n \geq m \\ n \in T}} K_n\right),$$

and

$$\bigcup_{\substack{n<m \\ n \in S}} K_n = \bigcup_{\substack{n<m \\ n \in T}} K_n = F.$$

If $x \in K \cup F$, then $d(x, F(T)) = 0$. If

$$x \in F(S) - (K \cup F) \subseteq \bigcup_{\substack{n \geq m \\ n \in S}} K_n,$$

then $d(x,F(T)) \leq d(x,K) \leq \rho(K_n,K)$ for some $n \geq m$. So if $x \in F(S)$, $d(x,F(T)) \leq \rho(K_n,K)$ for some $n \geq m$. Similarly, if $y \in F(T)$, $d(y,F(S)) \leq \rho(K_n,K)$ for some $n \geq m$. Thus

$$\delta(F(S),F(T)) \stackrel{def}{=} \textit{the Hausdorff distance of } F(S),\ F(T) \leq \sup_{n \geq m} \rho(K_n,K).$$

As $\rho(K_n,K) \to 0$ when $n \to \infty$, F is continuous.

Define now $\mathcal{J} \subseteq P(\mathbf{N})$ by

$$S \in \mathcal{J} \Leftrightarrow F(S) \in I.$$

Then $\mathcal{J}$ is G_δ in $2^{\mathbf{N}}$. Also $\mathcal{J}$ contains all the finite sets and thus $\mathcal{I} = \{\mathbf{N} - S : S \in \mathcal{J}\}$ contains all the cofinite sets. Also $\mathcal{I}, \mathcal{J}$ are G_δ, thus they are dense G_δ in $2^{\mathbf{N}}$. So $\mathcal{I} \cap \mathcal{J} \neq \emptyset$, i.e. there is $S \subseteq \mathbf{N}$ with S, $\mathbf{N} - S \in \mathcal{J}$ so that $F(\mathbf{N}) = F(S) \cup F(\mathbf{N} - S) = L \in I$. ■

Let now $I \subseteq K(E)$ be a G_δ ideal. Define the following derivative on $K(E)$:

$$K \mapsto K' = \{x \in K : \forall V \text{ open nbhd of } x : \overline{K \cap V} \notin I\}.$$

By iteration, define $K^{(\alpha)}$ by

$$K^{(0)} = K,$$

$$K^{(\alpha+1)} = (K^{(\alpha)})',$$

$$K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}, \quad \lambda \text{ limit}.$$

Then easily

$$K \in I_\sigma \qquad \text{iff } \exists \alpha < \omega_1 \ (K^{(\alpha)} = \emptyset).$$

Put for $K \in I_\sigma$,

$$|K| = \text{least } \alpha (K^{(\alpha)} = \emptyset).$$

We will show by induction on $|K|$ that

$$K \in I_\sigma \Rightarrow K \in I,$$

which completes the proof.

Assume it holds for all $K \in I_\sigma$, with $|K| < \alpha$. Fix then K with $|K| = \alpha$. Clearly, $\alpha = \beta + 1$ is a successor (unless $\alpha = 0$, in which case $K = \emptyset$ and we are done). As $K^{(\beta+1)} = \emptyset$, for every $x \in K^{(\beta)}$ there is an open nbhd V of x with $\overline{V \cap K^{(\beta)}} \in I$. For $n = 0,1,2,\ldots$ let V_n be an open nbhd of $K^{(\beta)}$ with $V_0 = E$, $\rho(\overline{V}_n, K^{(\beta)}) \leq 1/n$, if $n \geq 1$ and $V_n \supseteq V_{n+1}$. Put $L_n = K \cap (\overline{V}_n - V_{n+1})$. Then L_n is closed and $L_n^{(\gamma)} \subseteq K^{(\gamma)} \cap (\overline{V}_n - V_{n+1})$ for each γ, so $L_n^{(\beta)} = \emptyset$, i.e. $|L_n| < \alpha$ and so $L_n \in I$. Since clearly $\rho(L_n, K^{(\beta)}) \leq 1/n$, we have by the previous lemma that $K^{(\beta)} \cup (\bigcup_n L_n) = K \in I$ and the proof is complete. ∎

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